

Cosmological Models in the Generalized Einstein Action

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We have studied the evolution of the Universe in the generalized Einstein action of the form $R + \beta R^2$, where R is the scalar curvature and $\beta = \text{const.}$. We have found exact cosmological solutions that predict the present cosmic acceleration. These models also allow an inflationary de-Sitter era occurring in the early Universe. The cosmological constant (Λ) is found to decay with the Hubble constant (H) as, $\Lambda \propto H^4$. In this scenario the cosmological constant varies quadratically with the energy density (ρ), i.e., $\Lambda \propto \rho^2$. Such a variation is found to describe a two-component cosmic fluid in the Universe. One of the component accelerated the Universe in the early era, and the other in the present era. The scale factor of the Universe varies as $a \sim t^n$, $n = \frac{1}{2}$ in the radiation era. The cosmological constant vanishes when $n = \frac{4}{3}$ and $n = \frac{1}{2}$. We have found that the inclusion of the term R^2 mimics a cosmic matter that could substitute the ordinary matter.

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I. INTRODUCTION

Modified gravity models have been invoked to resolve cosmological and astrophysical problems with observations (Hawking, Luttrell, Whitt, Srivastava, Sinha, Kung). The generalized Einstein action including an additional scalar term R^2 is given by (Kenmoku et al., 1992) by

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} (R + 2\Lambda + \beta R^2) \quad (1)$$

where R : Ricci scalar curvature, Λ : the cosmological constant, g : the negative determinant of the metric tensor $g_{\mu\nu}$ and β is a constant. Several authors have studied classical solutions of this action without matter and have concluded that big bang singularity may be avoided (see Kung, 1996). In this paper we will study the cosmological implications of this action.

The variation of the metric with respect to $g_{\mu\nu}$ gives

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} + \beta B_{\mu\nu} = -8\pi G T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy momentum tensor of the cosmic fluid, and

$$B_{\mu\nu} = 2R(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) + 2(R_{;\mu\nu} - g_{\mu\nu}\Delta R), \quad (3)$$

with $R_{;\mu\nu} = \nabla_\mu \nabla_\nu R$ and $\Delta R = g^{\mu\nu} R_{;\mu\nu}$. For an ideal fluid one has

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (4)$$

where u_μ , ρ , p are the velocity, density and pressure of the cosmic fluid. The flat Robertson-Walker line element is given by

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \quad (5)$$

The time-time and space-space components of Eq.(2) give

$$3H^2 - \Lambda - 18\beta(6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2) = 8\pi G\rho, \quad (6)$$

and

$$-2\dot{H} - 3H^2 + \Lambda + 6\beta(2\ddot{H} + 12H\dot{H} + 18\dot{H}H^2 + 9\dot{H}^2) = 8\pi Gp, \quad (7)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble constant.

II. MODEL A

Now consider the cosmological model when

$$\Lambda = -18\beta(\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2) \quad (8)$$

so that Eq.(6)

$$3H^2 = 8\pi G\rho. \quad (9)$$

and Eq.(7) becomes

$$-2\dot{H} - 3H^2 = 8\pi G \left[p - \frac{3\beta}{2\pi G} (\ddot{H} + 3H\dot{H} + 12\dot{H}^2) \right], \quad (10)$$

A universe with bulk viscosity (η) is obtained by replacing the pressure p by the effective pressure $p - 3\eta H$. In this case, one may attribute that the inclusion of the R^2 is equivalent to having a bulk viscosity given by

$$\eta = \frac{\beta}{2\pi G} \left(\frac{\ddot{H}}{H} + 3\dot{H} + 12\frac{\dot{H}^2}{H} \right). \quad (11)$$

However for a power law expansion of the form

$$a = A t^n, \quad A, n \text{ const.} \quad (12)$$

one has $\ddot{H} + 3\dot{H}H = 0$, so that

$$\eta = \left(\frac{6\beta n}{\pi G} \right) t^{-3}. \quad (13)$$

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The cosmological constant then becomes

$$\Lambda = \left(\frac{54\beta}{n^3} \right) t^{-4}, \quad n \neq 0, \quad (14)$$

and the energy density

$$8\pi G\rho = \frac{3n^2}{t^2}. \quad (15)$$

Using Eq.(12) the cosmological constant becomes

$$\Lambda = \left(\frac{54\beta}{n} \right) H^4, \quad n \neq 0. \quad (16)$$

Upon using Eq.(9), this becomes

$$\Lambda = \frac{6\beta}{n^6} (8\pi G)^2 \rho^2, \quad n \neq 0. \quad (17)$$

Substituting Eq.(12) in Eq.(10), we see that the pressure is given by

$$8\pi p = \left(\frac{(2-3n)n}{t^2} + \frac{72\beta n(1-2n)}{t^4} \right). \quad (18)$$

Using Eq.(15), this can be written as

$$p = \left(\frac{2}{3n} - 1 \right) \rho - \left(\frac{1-2n}{n^3} \right) N \rho^2, \quad N = 64\pi G\beta \quad n \neq 0. \quad (19)$$

We know the van der Waals equation of state is given by

$$p = \frac{\gamma\rho}{1-b\gamma} - \alpha\rho^2, \quad \gamma, b, \alpha = \text{const.} \quad (20)$$

Thus, the resulting equation of state of a power law expansion is that due to two-component fluid resembling the van der Waals equation of state. Therefore, introducing a term of R^2 in the Einstein action is like introducing two fluid components in the Universe. We see that one component of the fluid drives the Universe in cosmic acceleration, by making $p < 0$, in some period and decelerates it in another period ($p > 0$). In the early Universe, when the density was so huge, p was negative if $n < \frac{1}{2}$. During the matter dominated epoch, when the density is very small, $p < 0$, if $n > \frac{2}{3}$. Hence, we see that the Universe accelerates for any deviation from the Einstein-de Sitter expansion.

A. Inflationary Era

We see from Eq.(9) when $H = H_0 = \text{const.}$, i.e., $a \propto \exp(H_0 t)$, the cosmological constant vanishes, i.e., $\Lambda = 0$. From Eq.(11) the bulk viscosity also vanishes, i.e., $\eta = 0$. We recover the de Sitter solution, $p = -\rho$ [Eqs.(9) and (10)]. For static universe $n = 0$, the cosmological constant, the bulk viscosity, the energy density and the pressure vanish, i.e., $\Lambda = \eta = p = \rho = 0$.

B. Radiation Dominated Era

During the radiation dominated phase, as in the Einstein-de Sitter model, i.e., $a \propto t^{\frac{1}{2}}$, one has $n = \frac{1}{2}$ so that Eq.(19) give the equation of the state $p = \frac{1}{3}\rho$. Thus the Einstein-de Sitter model is recovered. In this epoch the cosmological constant vanishes. However, Eq.(19) shows that any deviation from $n = \frac{1}{2}$ in the radiation era, viz., $n < \frac{1}{2}$, the second term will be large and negative. Thus, an accelerated expansion of the Universe will be inevitable.

C. Matter Dominated Era

In the matter dominated epoch of Einstein-de Sitter model one has $n = \frac{2}{3}$. In this case $p = 72\pi G\beta \rho^2$. Since ρ is small today, we see that the Universe asymptotically reduces to Einstein-de Sitter type. However, for any deviation of this expansion law, $n > \frac{2}{3}$ accelerated expansion will be inevitable. In this case $p < 0$. So, in the distant future, when $\rho \rightarrow 0$, the equation of state reduces to

$$p = \left(-1 + \frac{2}{3n} \right) \rho = \omega \rho \quad \left(\omega = -1 + \frac{2}{3n} \right). \quad (21)$$

Thus, $n > \frac{2}{3}$ implies $\omega > -1$. We remark here in the distant future, when $n \rightarrow \infty$, $p = -\rho$. Hence, the future of our Universe will be a de-Sitter expansion.

III. MODEL B

Now, let us define the cosmological constant by

$$\Lambda = -6\beta(2\ddot{H} + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2), \quad (22)$$

so that Eq.(6) and (7) become

$$3H^2 = 8\pi G(\rho + \bar{\rho}), \quad (23)$$

and

$$-2\dot{H} - 3H^2 = 8\pi Gp, \quad (24)$$

where

$$8\pi G\bar{\rho} = -12\beta(\ddot{H} + 3H\ddot{H} + 6\dot{H}^2) \quad (25)$$

A. Inflationary Era

We see that when $H = H_0 = \text{const.}$, i.e., $a \propto \exp(H_0 t)$, $p = -\rho$, $\Lambda = 0$ and $\bar{\rho} = 0$.

B. Radiation and Matter dominated Eras

Now consider a power law expansion of the scale factor of the form as in Eq.(12). We find

$$8\pi Gp = n(2-3n) t^{-2}, \quad (26)$$

$$\Lambda = 18\beta n(2n-1)(3n-4)t^{-4}, \quad (27)$$

and

$$\rho = \left(\frac{3n}{2-3n}\right)p + N' \left(\frac{2n-1}{n^2(2-3n)^2}\right)p^2, \quad N' = 637\beta G, \quad n \neq \frac{2}{3} \quad (28)$$

Eq.(28) represent our equation of state for the present cosmology. The cosmological constant here varies as

$$\Lambda = \frac{243}{2}\beta H^4, \quad (29)$$

The equation of state now reads,

$$p = \omega(t)\rho, \quad \omega(t) = \left(\frac{3n}{2-3n} + 72\beta \frac{(2n-1)}{n(2-3n)} \frac{1}{t^2}\right)^{-1}. \quad (30)$$

It is evident when $n \rightarrow \infty$ (i.e., $a \rightarrow \infty$), $\omega \rightarrow -1$ and the Universe becomes vacuum dominated and expands like de-Sitter. It is interesting to note that when $n = \frac{4}{3}$, the cosmological constant vanishes, i.e., $\Lambda = 0$. In this case the pressure becomes negative, i.e., $p < 0$, and this drives the Universe into an epoch of cosmic acceleration. The energy density becomes

$$\rho = c_1 H^2 + c_2 H^4, \quad c_1, c_2 \text{ const.} \quad (31)$$

The deceleration parameter $q = -\frac{\ddot{a}}{aH^2} = -0.25$. Once again, when $n = \frac{1}{2}$, we recover the Einstein-de Sitter solution, i.e., $a \propto t^{\frac{1}{2}}$ and $p = \frac{1}{3}\rho$ and $\Lambda = 0$. For $n = \frac{2}{3}$, $p = 0$, $\Lambda = -8\beta t^{-4}$, and $8\pi G\rho = \frac{4}{3t^2} \left(1 + \frac{12\beta}{t^2}\right)$. Hence, the Universe approaches the Einstein-de Sitter solution asymptotically ($t \rightarrow \infty$).

IV. MODEL C

Now consider a cosmological model in which $\Lambda = 0$. In this case, eqs.(6) and (7) yield

$$3H^2 = 8\pi G(\rho + \rho') \quad (32)$$

and

$$-2\dot{H} - 3H^2 = 8\pi G(p + p'), \quad (33)$$

where

$$8\pi G\rho' = 18\beta(6\dot{H}H^2 + 2H\ddot{H} - \dot{H}^2), \quad (34)$$

$$8\pi Gp' = -6\beta(2\ddot{H} + 12H\ddot{H} + 18\dot{H}H^2 + 9\dot{H}^2),$$

Eqs.(32) and (33) can be written as

$$3H^2 = 8\pi G\rho_{\text{eff.}} \quad (35)$$

and

$$-2\dot{H} - 3H^2 = 8\pi Gp_{\text{eff.}}, \quad (36)$$

where

$$\rho_{\text{eff.}} = \rho + \rho', \quad p_{\text{eff.}} = p + p'. \quad (37)$$

We, therefore, argue that the inclusion of the term R^2 in the Einstein action induces a fluid in the Universe that has pressure (p') and energy density (ρ'), in addition to the preexisting matter. This may suggest that our Universe is fluid with two components; one is *bright* (ρ) and the other is *dark* (ρ') without having a cosmological constant.

A. The Inflationary Era

An inflation solution arises when $H = H_0 = \text{const.}$ which is solved to give $a \propto \exp(H_0 t)$. Eq.(35) and (36) give

$$p_{\text{eff.}} = -\rho_{\text{eff.}} \quad (38)$$

With some scrutiny, one would discover that Eq.(34) implies that $p' = \rho' = 0$. Hence, the *dark* component does not contribute to this inflationary era.

B. Radiation Dominated Era

Consider now a power law expansion for the Universe during the radiation dominated era of the form

$$a = B t^n, \quad n, B, \text{ const.} \quad (39)$$

Substituting this in Eq.(34) one gets

$$\rho' = \frac{54\beta n^2(1-2n)}{8\pi G} t^{-4}, \quad (40)$$

and

$$p' = \frac{18\beta n(1-2n)(4-3n)}{8\pi G} t^{-4}. \quad (41)$$

The two equations are related by the relation

$$p' = \left(-1 + \frac{4}{3n}\right) \rho', \quad n \neq 0, \quad (42)$$

which represents the equation of state of the *dark* fluid. It is very interesting to note that, when $n = \frac{1}{2}$, $\rho' = p' = 0$. This implies that the *dark* component does not disturb the nucleosynthesis constraints set forth by the Einstein-de Sitter solution. Notice that when $n \rightarrow \infty$, $p' = -\rho'$, so that in the distant future the our Universe with or without normal matter will be vacuum dominated. We notice that this *dark* component does not live in a static Universe since it has $p' = \rho' = 0$. For a positive energy density, we have (for $\beta > 0$) the constraint $n < \frac{1}{2}$. For $\frac{1}{2} < n < \frac{4}{3}$, $\rho' < 0$, $p' < 0$. The positivity of energy density is recovered if $\beta < 0$. It is remarkable to notice that if the matter action is not incorporated in the Einstein action, the inclusion of the quadratic term acts like matter. This type of matter is characterized by its equation of state in Eq.(42). Hence, the inclusion of the R^2 mimics the introduction of a new matter in the Universe.

C. Matter Dominated Era

Applying Eqs.(39)-(41) into Eqs.(32) and (33), one gets

$$8\pi G\rho = \frac{3n^2}{t^2} \left(1 - \frac{18\beta(1-2n)}{t^2}\right), \quad (43)$$

and

$$8\pi Gp = \frac{n(2-3n)}{t^2} \left(1 - \frac{18\beta(2n-1)(3n-4)}{(2-3n)} \frac{1}{t^2}\right), \quad (44)$$

Once again, we see from Eq.(41) that the pressure of the *dark* fluid vanishes when $n = \frac{4}{3}$ leaving only the *bright* fluid to contribute to the universal pressure. In this case, $p < 0$, and this will drive the Universe into a cosmic acceleration era. Substitution of $n = \frac{2}{3}$ in Eqs.(43) and E(44) yields

$$8\pi G\rho = \frac{4}{3t^2} \left(1 + \frac{6\beta}{t^2}\right), \quad 8\pi Gp = \frac{8\beta}{t^4}. \quad (45)$$

We remark that the Universe approaches the Einstein-de Sitter asymptotically (when $t \rightarrow \infty$, i.e., in the distant future), where $\rho = \frac{1}{6\pi Gt^2}$ and $p = 0$. However, $\rho_{\text{eff.}} = \frac{1}{6\pi Gt^2}$ and $p_{\text{eff.}} = 0$. Notice that during this era the *dark* component behaves like a stiff matter, with $p' = \rho'$. It is evident that when $\frac{2}{3} < n < \frac{4}{3}$ the Universe enters a period of cosmic acceleration. We may thus argue that the present observed cosmic acceleration happened during this period. Eq.(45) looks like having a cosmological constant of the form $\Lambda \propto H^4$ in the Einstein-de Sitter universe.

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